By : Dir. Firoz Ahmad

MATHEMATICS

Mob. : 9470844028
9546359990

AIM POINTS
MATHEMATICS
DIR. FIROZ AHMAD

Ram Rajya More, Siwan

XIth, XIIth, TARGET IIT-JEE (MAIN + ADVANCE) & COMPETITIVE EXAM. FOR XII (PQRS)

BINARY OPERATIONS

& Their Properties

CONTENTS

THINGS TO REMEMBER

1. A binary operation on a set S is a function from S × S to S.
   A binary operation * on a set S associates any two elements a, b ∈ S to a unique element a * b ∈ S.
2. A binary operation * on a set S is said to be
   (i) commutative, if a * b = b * a for all a, b ∈ S.
   (ii) associative, if (a * b) * c = a * (b * c) for all a, b, c ∈ S.
   (iii) distributive over a binary operation o on S, if
   a * (b o c) = a(* b) o (a * c)
   and (b o c) * a = (b * a) o (c * a) for all a, b ∈ S
3. Let * be a binary operation on a set S. An element e ∈ S is said to be identity element for the binary operation *, if a * e = a = e * a for all a ∈ S.
4. Let * be a binary operation on a set S and e ∈ S be the identity element. An element a ∈ S is said to be invertible, if there exists on element b ∈ S such that
   a * b = e = b * a
5. A binary operation on a finite set can be completely described by means of composition table. From the composition table, we can infer the following properties of the binary operation :
   (i) The binary operation is commutative if the composition table is symmetric about the leading diagonal.
   (ii) If the row headed by an element say e coincides with row at the top and the column headed by e coincides with the column on the extreme left, then e is the identity element.
   (iii) If each row, except the top–most row, or each column, except the left–most column, contains the identity element. Then, every element of the set is invertible with respect to the given binary operation.
6. Total number of binary operations on a set consisting of n elements is n^{n^2}.

   Total number of commutative binary operations on a set consisting of n elements is $n^{\frac{n(n-1)}{2}}$.

EXERCISE-1

1. Let S be a non–empty set and P(S) be its power set. For any two subsets A and B of S, we know that A ∪ B ⊂ S. That is, for any two elements of P(S), we have A ∪ B ∈ P(S). Therefore, '∪' is a binary operation on P(S)
2. Let S = $\{a+\sqrt{2}b : a, b \in Z\}$. Then, prove that an operation * on S defined by

   $$\left(a_1+\sqrt{2}b_1\right)*\left(a_2+\sqrt{2}b_2\right)=\left(a_1+a_2\right)+\sqrt{2}\left(b_1+b_2\right)$$ fo all a_1, b_1, a_2, b_2 ∈ Z.

   is binary operation on S.
3. Let M be the set of all singular matrices of the form $\begin{bmatrix} x & x \\ x & x \end{bmatrix}$, where x is a non–zero real number.

   On M, let * be an operation on M.
4. Is * defined on the set {1, 2, 3, 4, 5} by a * b = LCM of a and b a binary operation ? Justify your answer.

5. Discuss the commutativity and associativity of the binary operation "*" on R defined by
   $a * b = a + b + ab$ for all $a, b \in R$
   there on RHS we have usual addition, subtraction and multiplication of real numbers.
6. Discuss the commutativity and associativity of the binary operation * on R defined by
   $a * b = \dfrac{ab}{4}$ for all a, b, $\in R$
7. Discuss the commutativity and associativity of binary operation '*' defined on Q by the rule
   $a * b = a - b + ab$ for all $a, b \in Q$.
8. Let A be a non–empty set and S be the set of all functions from A to itself. Prove that the composition of functions 'o' is a non–commutative binary operation on S. Also, prove that 'o' is an associative binary operation on S.
9. Let $A = N \times N$ and '*' be a binary operation on A defined by
   $(a, b) * (c, d) = (ac, bd)$ for all $a, b, c, d \in N$.
   Show that '*' is commutative and associative binary operation on A.
10. Let A be a set having more than one element. Let '*' be a binary operation on A defined by
    $a * b = a$ for all $a, b \in A$.
    Is '*' commutative or associative on A ?
11. Let '*' be a binary operation on N, the set of natural numbers, defined by
    $a * b = a^b$ for all $a, b \in N$.
    Is '*' associative or commutative on N ?
12. Let '*' be a binary operation on N given by
    $a * b = \text{HCF}(a, b), \quad a, b \in N$
    (i) Find : 12 * 4, 18 * 24, 7 * 5
    (ii) Check the commutativity and associativity of '*' on N.
13. Consider the binary operations $* : R * R \to R$ and $o : R \times R \to R$ defined as $a * b = |a - b|$ and $aob = a$ for all $a, b \in R$.
    Show that * is commutative but not associative, o is associative but not commutative. Further show that * is distributive over o. Does o distribute over * ? Justify your answer.
14. Determine which of the following binary operations are associative and which are commutative :
    (i) * on N defined by $a * b = 1$ for all $a, b \in N$
    (ii) * on Q defined by $a * b = \dfrac{a+b}{2}$ for all $a, b \in Q$.
15. Let '*' be a binary operation on a set S. If there exists an element $e \in S$ such that
    $a * e = a = e * a$ for all $a \in S$.
    Then, e is called an identity element for the binary operation '*' on set S.
16. Let '*' be a binary operation on a set S. If S has an identity element for '*', then it is unique.
17. If * is defined on the set R of all real numbers by $a * b = \sqrt{a^2 + b^2}$, find the identity element in R with respect to *.
18. Let '*' be an associative binary operation on a set S with the identity element e in S. Then, the inverse of an invertible element is unique.

19. Let * be an associative binary operation on a set S and a be an invertible element of S. Then, $(a^{-1})^{-1} = a$

20. On Q, the set of all rational numbers, a binary operation * is defined by

$$a * b = \frac{ab}{5} \text{ for all a, b} \in Q.$$

21. Let '*' be a binary operation on Q_0 (set of all non–zero rational numbers) defined by

$$a * b = \frac{ab}{4} \quad a, b \in Q_0.$$

Then, find the
(i) Identity element in Q_0
(ii) inverse of an element in Q_0.

22. Let X be a non–empty set and let '*' be a binary operation on P(X) (the power set of X) defined by
$A * B = A \cap B$ for A, B $\in$ P(X)

22. Let X be a non–empty set and let '*' be a binary operation on P(X) (the power set of set X) defined by
$A * B = (a - B) \cup (B - A)$ for all A, B $\in$ P(X)

23. Let A = Q × Q and let * be a binary operation on A defined by
$(a, b) * (c, d) = (ac, b + ad)$ for (a, b), (c, d) $\in$ A.
Then, with respect to * on A
(i) Find the identity element in A
(ii) Find the invertible elements of A.

24. Let $A = N \cup \{0\} \times N \cup \{0\}$ and let '*' be a binary operation on A defined by
$(a, b) * (c, d) = (a+ c, b + d)$ for (a, b), (c, d) $\in$ A.
Show that :
(i) '*' is commutative on A.
(ii) '*' is associative on A.

25. Show that the number of binary operations on {1, 2} having 1 as identity and having 2 as inverse of 2 is exactly one.

26. Determine the total number of binary operations on the set S = {1, 2}. Then, * is a function from S × S = {(1, 1), (1, 2), (2, 1), (2, 2)} to S = {1, 2}.

27. Let 'o' be a binary operation on the set Q_0 of a non–zero rational numbers defined by

$$a \text{ o } b = \frac{ab}{2}, \text{ for all a, b,} \in Q_0.$$

(i) Show that 'o' is both commutative and associate.
(ii) Find the identity element in Q_0.
(iii) Find the invertible elements of Q_0.

28. Let * be a binary operation on Z defined by
$a * b = a + b - 4$ for all a, b, $\in$ Z
(i) Show that 'o' is both commutative and associative.
(ii) Find the identity element in Z.
(iii) Find the invertible elements of Z.

29. Let * be the binary operation on N defined by
    a * b = HCF of a and b.
    Does there exist identity for this binary operation one N ?

30. Let * be a binary operation on Q_0 (set of non–zero rational numbers) defined by

    $$a * b = \frac{3ab}{5} \text{ for all a, b} \in Q_0.$$

31. Consider the set S = {1, –1} of square roots of unity and multiplication (×) as a binary operation on S. Construct the composition table for multiplication (×) on S. Also, find the identity element for multiplication on S and the inverses of various elements.

32. Consider the set S = {1, –1, i, –i} of fourth roots of unity. Construct the composition table for multiplication on S and deduce its various properties.

33. Consider the set S = {1, 2, 3, 4}. Define a binary operation * on S as follows :
    a * b = r, where r is the least non–negative remainder when ab is divided by 5.
    Construct the composition table fo '*' on S.

34. Construct the composition table for the composition of functions (o) defined on the set S = {f_1, f_2, f_3, f_4} of four functions from C (the set of all complex numbers) to itself, defined by

    $$f_1(z) = z,\ f_2(z) = -z,\ f_3(z) = \frac{1}{z},\ f_4(z) = -\frac{1}{z} \text{ for all } z \in C.$$

35. Consoder the infimum binary operation ^ on the set S = {1, 2, 3, 4, 5} defined by
    a ^ b = Minimum of a and b.
    Write the composition table of the operation ^.

36. Consider a binary operation * on the set {1, 2, 3, 4, 5} given by the following multiplication table

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

    (i) Compute (2 * 3) * 4 and 2 * (3 * 4)
    (ii) Is * commutative ?
    (iii) Compute (2 * 3) * (4 * 5)

37. Define a binary operation * on the set A = {0, 1, 2, 3, 4, 5} as
    a * b = a + b (mod 6)
    Show that zero is the identity for this operation and each element a of the set is invertible with 6 – a being the inverse of a.

38. Define a binary operation * on the set {0, 1, 2, 3, 4, 5} as

    $$a * b = \begin{cases} a+b, & \text{if } a+b<6 \\ a+b-6, & \text{if } a+b \geq 6 \end{cases}$$

    Show that 0 is the identity for this operation and each element a ≠ 0 of the set is invertible with 6 – a being the inverse of a.

39. Define a binary operation of 4.

40. Write the identity element for the binary operation * defined on the set R of all real numbers by the rule

$$a * b = \frac{3ab}{7} \text{ for all } a, b \in R$$

41. Let * be a binary operation, on the set of all non–zero real numbers, given by

$$a * b = \frac{ab}{5} \text{ for all } a, b \in R - \{0\}$$

42. Write the composition table for the binary operation multiplication modulo 10 ($\times_{10}$) on the set S = {2, 4, 6, 8}

43. Let * be a binary operation defined by a * b = 3a + 4b – 2. Find 4 * 5.

44. Let * be a binary operation on N given by a * b = HCF (a, b), a, b ∈ N. Write the value of 22 * 4.

EXERCISE-3

1. An operation * is defined on the set Z of non–zero integers by $a * b = \frac{a}{b}$ for all a, b ∈ Z. Then the property satisfied is
   (a) closure (b) commutative (c) associative (d) None of these

2. Let * be a binary operation on Q^+ defined by $a * b = \frac{ab}{100}$ for all a, b ∈ Q^+. The inverse of 0.1 is
   (a) 10^5 (b) 10^4 (c) 10^6 (d) non–existent

3. Consider the binary operation * defined on Q – {1} by the rule
   a * b = a + b – ab for all a, b, ∈ Q – {1}
   The identity element in Q – {1} is
   (a) 0 (b) 1 (c) $\frac{1}{2}$ (d) –1

4. For the multiplication of matrices as a binary operation on the set of all matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, a, b ∈ R the inverse of $\begin{bmatrix} 2 & 4 \\ -3 & 5 \end{bmatrix}$ is
   (a) $\begin{bmatrix} -2 & 3 \\ -3 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2/13 & 3/13 \\ 3/13 & 2/13 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

5. On the set Q^+ of all positive rational numbers a binary operation * is defined by $a * b = \frac{ab}{2}$ for all a, b, ∈ Q^+. The inverse of 8 is
   (a) $\frac{1}{8}$ (b) $\frac{1}{2}$ (c) 2 (d) 4

6. The number of binary operations that can be defined on a set of 2 elements is
   (a) 8 (b) 4 (c) 16 (d) 64

7. For the binary operation * on Z defined by a * b = a + b + 1 the identity element is
   (a) 0 (b) –1 (c) 1 (d) 2